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Irreducible Representations of Power-associative Train Algebras*

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Abstract. Train algebras were introduced by Etherington in 1939 as an algebraic framework for treating genetic problems. The aim of this paper is to study the representations and irreducible representations of power-associative train algebras of rank 4. There are three families of such algebras and for two of them we prove that every irreducible representation has dimension one over the ground field. For the third family we give an example of an irreducible representation of dimension three.

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1 Introduction

Let F be an infinite field of characteristic $\neq 2$ and A a commutative not necessarily associative algebra over F. A pair (A, w) is called a *baric algebra* if $w : A \to F$ is a nonzero algebra homomorphism. A baric algebra (A, w) is called a *train algebra* of rank t if it satisfies an equation of the form $x^t + \alpha_1 w(x) x^{t-1} + \cdots + \alpha_{t-1} w(x)^{t-1} x = 0$, where t is the minimal positive integer for which the identity holds, $\alpha_1, \ldots, \alpha_{t-1}$ are fixed elements in F and x^t is the t-th principal power of x defined by $x^1 = x$ and $x^{k+1} = x^k x$ for $k \geq 1$. The above equation is called the *train equation* of A. Train algebras were introduced by Etherington [5, 6] as an algebraic framework for treating genetic problems.

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In this work we deal with representations of algebras. Let A be an algebra which belongs to a class $\mathcal{C}$ of algebras over F and let M be a vector space over F. Following Eilenberg [4], a linear map $\mu : A \to \mathrm{End}(M)$ is called a representation of A in the class $\mathcal{C}$ if the split null extension $\overline{A} = A \oplus M$ of M with multiplication given by $(a+m)(b+n) = ab + \mu(a)(n) + \mu(b)(m)$ for all $a, b \in A$ and $m, n \in M$ belongs to the class $\mathcal{C}$. Let A be a train algebra of rank t, M a vector space over F and $\mu : A \to \mathrm{End}(M)$ a linear map. We will say that μ is a *train representation of A of rank t* if the split null extension $A \oplus M$ with homomorphism $\overline{w}$ defined by $\overline{w}(a+m) = w(a)$ for all $a \in A$ and $m \in M$ is a train algebra of rank t. A representation $\mu : A \to \mathrm{End}(M)$ is said to be *irreducible* if $M \neq 0$ and there is no proper subspace of M which is invariant under all the transformations $\mu(a)$ for $a \in A$, and is said to be *r-dimensional* if $\dim M = r$.

The aim of this paper is to study the representations and irreducible A-modules of power-associative train algebras of rank 4, following the ideas given by Bernad, Iltyakov and Martínez [3]. In [8] Labra and Reyes gave a characterization of the representations of train algebras of rank 3, and proved that every irreducible module has dimension one over F. In [9] López-Sánchez and Rodríguez Santa María proved that there are three families of power-associative train algebras of rank 4. We will use this classification to prove that for two of these families, every irreducible module has dimension one over F, and we show with an example that the third family may have irreducible representations of larger dimension.

2 Representations of Power-associative Train Algebras of Rank 4

Let A be a train algebra of rank 4, that is, a baric algebra satisfying

$$x^4 - (1+\gamma_1+\gamma_2)w(x)x^3 + (\gamma_1+\gamma_2+\gamma_1\gamma_2)w(x)^2x^2 - \gamma_1\gamma_2 w(x)^3 x = 0, \tag{1}$$

where $\gamma_0 = 1, \gamma_1, \gamma_2$ are the principal train roots of A. Since the elements $x + m \in A \oplus M$ satisfy the identity (1), we have

Lemma 2.1. *Let A be a baric algebra satisfying* (1) *and $\mu : A \to \mathrm{End}(M)$ a linear map. Then μ is a train representation of A of rank 4 if and only if for every $x \in A$,*

$$\begin{gathered}\mu(x^3) + \mu(x)\mu(x^2) + 2\mu(x)^3 - (1+\gamma_1+\gamma_2)w(x)\mu(x^2) \\ -2(1+\gamma_1+\gamma_2)w(x)\mu(x)^2 + 2(\gamma_1+\gamma_2+\gamma_1\gamma_2)w(x)^2\mu(x) - \gamma_1\gamma_2 w(x)^3 I = 0.\end{gathered} \tag{2}$$

An algebra A is power-associative if the subalgebra generated by any element x in A is associative. It is well known that if $\mathrm{char}(F) \neq 2, 3, 5$ then power-associativity is equivalent to the identity $x^2x^2 = x^4$.

In the following let F be an infinite field of characteristic $\neq 2, 3, 5$. Since the elements $x + m \in A \oplus M$ satisfy the identity $(x+m)^4 = (x+m)^2(x+m)^2$, we have

Lemma 2.2. *Let A be a power-associative algebra over F and let $\mu : A \to \mathrm{End}(M)$ be a linear map. Then μ is a representation of A in the class of power-associative algebras if and only if for every $x \in A$, we have*

$$\mu(x^3) + \mu(x)\mu(x^2) + 2\mu(x)^3 - 4\mu(x^2)\mu(x) = 0. \tag{3}$$

López-Sánchez and Rodríguez Santa María [9] proved that a power-associative train algebra of rank 4 satisfies one of the following identities:

$$x^4 - w(x)x^3 = 0, \tag{4}$$

$$x^4 - 3w(x)x^3 + 3w(x)^2x^2 - w(x)^3x = 0, \tag{5}$$

$$x^4 - 2w(x)x^3 + w(x)^2x^2 = 0. \tag{6}$$

We note that algebras satisfying (4) or (5) are Jordan algebras (see [1] and [10]). Moreover as the following examples show, there are algebras satisfying (6) which are not Jordan algebras.

Example 2.3. [11] Let A be a commutative real algebra with base $\{e, u, z_1, z_2\}$ and multiplication table given by $e^2 = e$, $eu = \frac{1}{2}u$, $ez_1 = z_1$, $uz_1 = z_2$ and other products being zero. Let $w : A \to \mathbb{R}$ be given by $w(e) = 1$ and $w(u) = w(z_1) = w(z_2) = 0$. Then (A, w) is a train algebra of rank 4 satisfying $x^4 - 2w(x)x^3 + w(x)^2x^2 = 0$. But A is power-associative since $x^4 = x^2x^2$, and it is not a Jordan algebra since if $x = e + u$ and $y = e + u + z_1 + z_2$ then $(x^2y)x \neq x^2(yx)$.

Example 2.4. [2] Let A be a commutative real algebra with base $\{e,u_1,u_2,u_3,u_4,s,t\}$ and multiplication table given by $e^2 = e$, $eu_i = \frac{1}{2}u_i$ $(i = 1, 2, 3, 4)$, $et = t$, $u_1u_4 = -s - t$, $u_1s = u_1t = u_3$, $u_2u_3 = s + t$, $u_2s = u_2t = u_4$ and other products being zero. Let $w : A \to \mathbb{R}$ be given by $w(e) = 1$ and $w(u_i) = w(s) = w(t) = 0$. Then (A, w) is a train algebra of rank 4 satisfying $x^4 - 2w(x)x^3 + w(x)^2x^2 = 0$. But A is power-associative since $x^4 = x^2x^2$, and it is not a Jordan algebra since if $x = u_1 + s$ and $y = u_2$ then $(x^2y)x = 4u_3 \neq x^2(yx) = 0$.

Now first we will consider algebras satisfying the identity (4) or (5), that is, $x^4 - (1+2\lambda)w(x)x^3 + \lambda(2+\lambda)w(x)^2x^2 - \lambda^2w(x)^3x = 0$ with $\lambda = 0$ or $\lambda = 1$. In [10] the authors proved that these algebras always have an idempotent element e and they also proved that the Peirce decomposition of A is $A = Fe \oplus A_{\frac{1}{2}} \oplus A_\lambda$, where $A_i = \{x \in \ker(w) \,|\, ex = ix\}$ and the following relations are fulfilled:

$$(A_{\frac{1}{2}})^2 \subseteq A_\lambda, \quad A_{\frac{1}{2}}A_\lambda \subseteq A_{\frac{1}{2}}, \quad A_\lambda^2 \subseteq A_\lambda. \tag{7}$$

Our aim is to show that every irreducible representation of such an algebra is one-dimensional. Let $\mathcal{C}_\lambda$ be the class of power-associative algebras satisfying the identity $x^4 - (1 + 2\lambda)w(x)x^3 + \lambda(2 + \lambda)w(x)^2x^2 - \lambda^2w(x)^3x = 0$ with $\lambda \in \{0, 1\}$.

Proposition 2.5. *Let A be an algebra in $\mathcal{C}_\lambda$, let $e \in A$ be an idempotent and let $\mu : A \to \mathrm{End}(M)$ be a finite-dimensional representation in the class $\mathcal{C}_\lambda$. Then $B = \langle \mu(a) \,|\, a \in \ker(w) \rangle$ the subalgebra of $\mathrm{End}(M)$ generated by $\mu(\ker(w))$ is nilpotent.*

Proof. Since A satisfies (4) or (5), $A \oplus M$ also satisfies one of those identities and is therefore Jordan. This also implies that $\ker(\overline{w})$ is nilpotent. Let n be the nilpotency index of $\ker(\overline{w})$ so that any product of n elements in $\ker(\overline{w})$, regardless of the association type, is zero. For any $a_1, \ldots, a_{n-1} \in \ker(w)$ and for every $m \in M$, $(a_1, 0)((a_2, 0)(\cdots((a_{n-1}, 0)(0, m))\cdots)) = (0, 0)$. On the other hand,

$$(a_1, 0)((a_2, 0)(\cdots((a_{n-1}, 0)(0, m))\cdots) = (0, \mu(a_1)\mu(a_2)\cdots\mu(a_{n-1})m).$$

Since $m \in M$ is arbitrary, we conclude that any product of $n-1$ generators of B is zero and therefore $\mu(\ker(w))^{n-1} = 0$. Notice that $B = \sum_{i=1}^{\infty} \left(\mu(\ker(w))\right)^i$ and $B^k = \sum_{i=k}^{\infty} \left(\mu(\ker(w))\right)^i$, and thus $B^{n-1} = 0$. □

In what follows, we will write $a \cdot m$ instead of $\mu(a)(m)$.

Lemma 2.6. *Let A be an algebra in the class $\mathcal{C}_\lambda$, let $e \in A$ be an idempotent and let $\mu : A \to \mathrm{End}(M)$ be a representation in $\mathcal{C}_\lambda$. Then $M = M_{\frac{1}{2}} \oplus M_\lambda$, where $M_i = \{m \in M \mid e \cdot m = im\}$ and the action of A on M satisfies*

$$\begin{gathered} e \cdot M_{\frac{1}{2}} = M_{\frac{1}{2}}, \quad e \cdot M_\lambda \subseteq M_\lambda, \quad A_\lambda \cdot M_\lambda \subseteq M_\lambda, \\ A_{\frac{1}{2}} \cdot M_\lambda \subseteq M_{\frac{1}{2}}, \quad A_\lambda \cdot M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}}, \quad A_{\frac{1}{2}} \cdot M_{\frac{1}{2}} \subseteq M_\lambda. \end{gathered} \tag{8}$$

Moreover if $x \in A_\lambda$ and $y \in A_{\frac{1}{2}}$, then

$$4(2\lambda - 1)\left[\mu(x)\mu(e) - \mu(e)\mu(x)\right] + 4(\lambda - \lambda^2)\mu(x) = 0, \tag{9}$$

$$-4(2\lambda - 1)\left[\mu(y)\mu(e) + \mu(e)\mu(y)\right] + (4\lambda^2 + 4\lambda - 2)\mu(y) = 0, \tag{10}$$

$$8\mu(xy)\mu(e) - 2(1+2\lambda)\mu(x)\mu(y) + 2(1-2\lambda)\mu(y)\mu(x) - 2(1+2\lambda)\mu(xy) = 0. \tag{11}$$

Proof. Subtracting (3) from (2) with $\gamma_1 = \gamma_2 = \lambda$, we get

$$\begin{gathered} 4\mu(x^2)\mu(x) - (1+2\lambda)w(x)\mu(x^2) - 2(1+2\lambda)w(x)\mu(x)^2 \\ + 2\lambda(2+\lambda)w(x)^2\mu(x) - \lambda^2 w(x)^3 I = 0. \end{gathered} \tag{12}$$

Replacing $x = e$ in (12) we get $((1-2\lambda)\mu(e) + \lambda^2 I)(2\mu(e) - I) = 0$. For $\lambda = 0$ this is $\mu(e)(2\mu(e) - I) = 0$, and for $\lambda = 1$ it is $(-\mu(e) + I)(2\mu(e) - I) = 0$, proving the stated decomposition of M. Now (8) follows from (7). Linearizing (12) we have

$$\begin{gathered} 8(\mu(xy)\mu(z) + \mu(xz)\mu(y) + \mu(yz)\mu(x)) \\ - 2(1+2\lambda)(w(x)\mu(yz) + w(y)\mu(xz) + w(z)\mu(xy)) \\ - 2(1+2\lambda)(w(x)\mu(y)\mu(z) + w(x)\mu(z)\mu(y) + w(y)\mu(x)\mu(z) \\ + w(y)\mu(z)\mu(x) + w(z)\mu(x)\mu(y) + w(z)\mu(y)\mu(x)) \\ + 4\lambda(2+\lambda)(w(xy)\mu(z) + w(xz)\mu(y) + w(yz)\mu(x)) - 6\lambda^2 w(xyz) = 0. \end{gathered} \tag{13}$$

Considering $x \in A_\lambda$ and $y = z = e$, (13) becomes (9). Considering $y \in A_{\frac{1}{2}}$ and $x = z = e$, (13) becomes (10). And considering $x \in A_0$, $y \in A_{\frac{1}{2}}$ and $z = e$, (13) becomes (11). □

Theorem 2.7. *Let A be an algebra in the class $\mathcal{C}_\lambda$. Then every finite-dimensional irreducible representation of A in $\mathcal{C}_\lambda$ is one-dimensional.*

Proof. Let $L_0 = \{m \in M \mid A_{\frac{1}{2}} \cdot m = 0\}$. Clearly, $A_{\frac{1}{2}} \cdot L_0 = 0$. Let $x \in A_\lambda$, $y \in A_{\frac{1}{2}}$ and $m \in L_0$. By (10), we have $y \cdot (e \cdot m) = -e \cdot (y \cdot m) + \frac{2\lambda^2 + 2\lambda - 1}{2(2\lambda - 1)} y \cdot m = 0$, so that $e \cdot L_0 \subseteq L_0$. By (11) and since $xy \in A_\lambda A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$, we have

$$y \cdot (x \cdot m) = \tfrac{1}{1-2\lambda}\big((1+2\lambda)(xy) \cdot m + (1+2\lambda)x \cdot (y \cdot m) - 4(xy) \cdot (e \cdot m)\big) = 0,$$

so that $A_\lambda \cdot L_0 \subseteq L_0$ and L_0 is a submodule of M. Since $\mu(A_{\frac{1}{2}}) \subseteq \mu(\ker(w))$, we can use Proposition 2.5 to show that $L_0 \neq 0$ and since M is irreducible, $L_0 = M$.

Let $L_1 = \{m \in M \mid A_\lambda \cdot m = 0\}$. Clearly, $A_{\frac{1}{2}} \cdot L_1 = 0$ and $A_\lambda \cdot L_1 = 0$. Let $x \in A_\lambda$ and $m \in L_1$. By (9), $x \cdot (e \cdot m) = e \cdot (x \cdot m) - \frac{\lambda - \lambda^2}{2\lambda - 1} x \cdot m = 0$, so L_1 is a submodule of M. Since $\mu(A_0) \subseteq \mu(\ker(w))$, we can use Proposition 2.5 to show that $L_1 \neq 0$ and since M is irreducible, $L_1 = M$.

Thus only $\mu(e) \neq 0$, so M is one-dimensional. □

Corollary 2.8. *Let A be a finite-dimensional algebra in the class $\mathcal{C}_\lambda$. Then the only irreducible representations of A in $\mathcal{C}_\lambda$ are $\mu_1(x) = \frac{1}{2}w(x)$ and $\mu_2(x) = \lambda w(x)$ (when $\lambda = 1$).*

Now we will consider algebras satisfying the identity (6). Let A be a power-associative algebra satisfying $x^4 - 2w(x)x^3 + w(x)^2x^2 = 0$. According to [7], A has an idempotent element e and the Peirce decomposition of A is $A = Fe \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_1$, where $A_i = \{x \in \ker(w) \mid ex = ix\}$. Moreover, these subspaces satisfy the relations $A_0^2 = A_1^2 = A_0A_1 = 0$, $A_{\frac{1}{2}}A_0 \subseteq A_{\frac{1}{2}} \oplus A_1$, $A_{\frac{1}{2}}A_1 \subseteq A_{\frac{1}{2}} \oplus A_0$ and $A_{\frac{1}{2}}^2 \subseteq A_0 + A_1$.

Lemma 2.9. *Let A be a train algebra satisfying $x^4 - 2w(x)x^3 + w(x)^2x^2 = 0$ and $e \in A$ be an idempotent. Let $\mu : A \to \mathrm{End}(M)$ be a representation in the corresponding class so that μ satisfies (2) with $\gamma_1 = 0$ and $\gamma_2 = 1$, i.e.,*

$$\mu(x^3) + \mu(x)\mu(x^2) + 2\mu(x)^3 - 2w(x)\mu(x^2) - 4w(x)\mu(x)^2 + 2w(x)^2\mu(x) = 0 \quad (14)$$

for all $x \in A$. Then $M = M_{\frac{1}{2}} \oplus M_0 \oplus M_1$, where $M_i = \{m \in M \mid e \cdot m = im\}$ and the action of A on M satisfies $e \cdot M_{\frac{1}{2}} = M_{\frac{1}{2}}$, $e \cdot M_0 = 0$, $e \cdot M_1 = M_1$, $A_0 \cdot M_0 = A_1 \cdot M_1 = A_0 \cdot M_1 = A_1 \cdot M_1 = 0$, $A_{\frac{1}{2}} \cdot M_0 \subseteq M_{\frac{1}{2}} \oplus M_1$, $A_{\frac{1}{2}} \cdot M_1 \subseteq M_{\frac{1}{2}} \oplus M_0$, $A_{\frac{1}{2}} \cdot M_{\frac{1}{2}} \subseteq M_0 \oplus M_1$, $A_0 \cdot M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_1$ and $A_1 \cdot M_{\frac{1}{2}} \subseteq M_0 \oplus M_{\frac{1}{2}}$. Moreover if A is power-associative, then for all $x \in A$,

$$2\mu(x^2)\mu(x) - 2w(x)\mu(x)^2 - w(x)\mu(x^2) + w(x)^2\mu(x) = 0, \quad (15)$$

which factors as $(\mu(x^2) - w(x)\mu(x))(2\mu(x) - w(x)I) = 0$.

Proof. Replacing $x = e$ in (14), we get $\mu(e)(\mu(e) - I)(2\mu(e) - I) = 0$, proving the stated decomposition of M. Subtracting (3) from (14) and dividing by 2, we get (15). □

Remark. Jordan train algebras may have irreducible representations of dimension more than one in the class of train algebras satisfying the identity (6).

Example 2.10. Consider a commutative real algebra A with base $\{e, u\}$, multiplication table given by $e^2 = e$, $eu = \frac{1}{2}u$ and $u^2 = 0$, and $w : A \to F$ given by $w(e) = 1$ and $w(u) = 0$. Moreover, A satisfies the identity $x^2(yx) = (x^2y)x$ and A is a Jordan algebra satisfying (6). Let M be a three-dimensional space. We will show an irreducible representation for A.

For $x = a_1e + a_2u \in A$, set $\mu(x) = \begin{pmatrix} 0 & a_2 & a_2 \\ 2a_2 & \frac{1}{2}a_1 & a_2 \\ -4a_2 & 2a_2 & a_1 \end{pmatrix}$. Then μ determines a three-dimensional representation. We have $\mu(x^2) = a_1\mu(x)$, $\mu(x^3) = a_1^2\mu(x)$, and

$$\begin{aligned} &2\mu(x)^3 + \mu(x)\mu(x^2) + \mu(x^3) - 4w(x)\mu(x)^2 - 2w(x)\mu(x^2) + 2w(x)^2\mu(x) \\ =\ &2\mu(x)^3 + a_1\mu(x)^2 + a_1^2\mu(x) - 4a_1\mu(x)^2 - 2a_1^2\mu(x) + 2a_1^2\mu(x) \end{aligned}$$

$$= 2\mu(x)^3 - 3a_1\mu(x)^2 + a_1^2\mu(x) = (2\mu(x) - a_1 I)(\mu(x) - a_1 I)\mu(x)$$

$$= \begin{pmatrix} -a_1 & 2a_2 & 2a_2 \\ 4a_2 & 0 & 2a_2 \\ -8a_2 & 4a_2 & a_1 \end{pmatrix} \begin{pmatrix} -a_1 & a_2 & a_2 \\ 2a_2 & -\frac{1}{2}a_1 & a_2 \\ -4a_2 & 2a_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_2 & a_2 \\ 2a_2 & \frac{1}{2}a_1 & a_2 \\ -4a_2 & 2a_2 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1^2 - 4a_2^2 & -2a_1a_2 + 4a_2^2 & -a_1a_2 + 2a_2^2 \\ -4a_1a_2 - 8a_2^2 & 8a_2^2 & 4a_2^2 \\ 4a_1a_2 + 8a_2^2 & -8a_2^2 & 4a_2^2 \end{pmatrix} \begin{pmatrix} 0 & a_2 & a_2 \\ 2a_2 & \frac{1}{2}a_1 & a_2 \\ -4a_2 & 2a_2 & a_1 \end{pmatrix} = 0.$$

Moreover,

$$2\mu(x)^3 + \mu(x)\mu(x^2) + \mu(x^3) - 4\mu(x^2)\mu(x)$$
$$= 2\mu(x)^3 + a_1\mu(x)^2 + a_1^2\mu(x) - 4a_1\mu(x)^2 = 2\mu(x)^3 - 3a_1\mu(x)^2 + a_1^2\mu(x) = 0.$$

Therefore, μ is a representation of A in the class of power-associative algebras satisfying the identity $x^4 - 2w(x)x^3 + w(x)^2x^2 = 0$.

The representation is clearly irreducible since an invariant submodule of M would have to be invariant under $\mu(e)$, so it could only be generated by some of the canonical basis vectors of M. These can be easily checked.

Problem. Describe the class of finite-dimensional power-associative train algebras satisfying the identity (6), which have faithful irreducible representations.

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